On the embedding of branes in five-dimensional spaces

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Abstract

We investigate the embedding of n-dimensional branes in $(n+1)$-dimensional spaces. We firstly consider the case when the embedding space is a vacuum bulk whose energy-momentum tensor consists of a Dirac delta function with support in the brane. We then consider the embedding problem in the context of Randall-Sundrum-type models, taking into account Z_2 symmetry and a cosmological constant. We employ the Campbell-Magaard theorem to construct the embeddings and are led to the conclusion that the content of energy-matter of the brane does not necessarily determine its curvature. Finally, as an application to illustrate our results, we construct the embedding of four-dimensional Minkowski spacetime filled with dust.

Higher-dimensional theories of gravity and, in particular, the recent braneworld scenario [1–5], in which our visible (under TeV scale of energy) spacetime appears as a hypersurface embedded in a higher dimensional space, have greatly contributed to bring about a renewed interest in both mathematical and physical aspects of embedding theories of spacetime [6–21]. As far as their mathematical structure is concerned embedding theories are naturally subject to the theorems of differential geometry. These theorems, in turn, may give us deeper insights on our understanding of the relationship between the geometry of the lower-dimensional world and that of the embedding space. In this respect, an important theorem due to Campbell and Magaard, along with its extended versions, may shed some light on the mathematical structure of five-dimensional non-compactified Kaluza-Klein gravity theories [11] as well as Randall-Sundrum (RS) models [4,5]. Local isometric embeddings of Riemannian manifolds have long been studied in differential geometry. Of particular interest is a well known theorem (Janet-Cartan) [22,23] which states that if the embedding space is flat, then the minimum number of extra dimensions needed to analytically embed a n-dimensional Riemannian manifold is d, with $0 \leq d \leq n(n-1)/2$. The novelty brought by Campbell-Magaard [24,25] theorem is that the number of extra dimensions falls drastically

to $d = 1$ when the embedding manifold is allowed to be Ricci-flat (instead of Riemann-flat). It is worth mentioning that the Campbell-Magaard theorem can be extended to more general contexts, such as those in which the higher-dimensional space is sourced by a cosmological constant or by a scalar field and even by an arbitrary non-degenerate Ricci tensor [26–29]. In this letter we are interested in the case when the embedding higher-dimensional space has a source which is singular and corresponds to a Dirac delta function whose support is the four dimensional spacetime. Clearly, this question is motivated by the Randall-Sundrum models, according to which the standard particle model is confined to our four dimensional spacetime, and hence the matter-energy content of the spacetime is represented by a delta term. Some aspects of this problem have been studied recently [14,17,19,21], mainly in connection with the Z_2-symmetry involved in the Randall-Sundrum models.

In this paper we are concerned with general aspects of the existence problem of embeddings, in the same direction of the Campbell-Magaard theorem. More specifically, we are interested in knowing what are the embedding conditions when the energy-momentum tensor of the embedding space is singular. In other words, we wonder if any n-dimensional spacetime with arbitrary metric and energy-momentum tensor can be embedded in a $(n+1)$-dimensional vacuum space. We also discuss the embedding problem in the context of Randall-Sundrum type models.

Let us start our discussion with a brief review of the Campbell-Magaard theorem, which states that any n dimensional space can be analytically and isometrically embedded in a vacuum space of $n+1$ dimensions [24,25]. We recall that in the proof provided by Campbell and Magaard the embedding problem is reduced to an initial value one. An outline of their proof will be briefly given as follows. Consider the metric of the $n + 1$ space written in a Gaussian coordinate system

$$ds^2 = \bar{g}_{ij}\left(x, y\right) dx^i dx^j + dy^2, \tag{1}$$

where $x = (x^1, ..., x^n)$, and Latin indices run from 1 to n while Greek ones go from 1 to $n + 1$.

It is easy to verify that the $(n+1)$-dimensional Einstein vacuum equations $R_{\mu\nu} = 0$ expressed in the above coordinates are equivalent to the following set of equations[1]:

$$\bar{R}_{ij} + \left(\overline{K}_{ij}\overline{K} - 2\overline{K}_{im}\overline{K}_j^m\right) - \frac{\partial \overline{K}_{ij}}{\partial y} = 0 \tag{2}$$

$$\bar{\nabla}_j \left(\overline{K}^{ij} - \bar{g}^{ij}\overline{K}\right) = 0 \tag{3}$$

$$\bar{R} + \overline{K}^2 - \overline{K}_{ij}\overline{K}^{ij} = 0, \tag{4}$$

where $\bar{\nabla}_j$ is the covariant derivative with respect to $\bar{g}_{ik}$; $\bar{R}_{ik}$, $\bar{R}$, and $\overline{K}_{ik}$ denote, respectively, the Ricci tensor, the scalar curvature and the extrinsic curvature of the hypersurface $y = y_0 = const$, which has an induced metric given by $\bar{g}_{ij}(x, y_0)$, and $\overline{K}$ is the trace of $\overline{K}_{ik}$. Recall that in the Gaussian coordinates adopted the extrinsic curvature assumes the simple form:

$$\overline{K}_{ij} = -\frac{1}{2}\frac{\partial \bar{g}_{ij}}{\partial y}. \tag{5}$$

Owing to the Bianchi identities not all of the equations (2), (3) and (4) are independent. In fact, we can show that the first one propagates the others in the sense that, if Eqs. (3) and (4) are satisfied in a hypersurface $y = 0$, for example, and equation (2) is valid in an open set of the $(n+1)$-dimensional space, then Eqs. (3) and (4) will be satisfied in a certain open set of the $(n+1)$-dimensional space. Therefore, it is sufficient to demand that the constraint equations (3) and (4) be satisfied in the hypersurface $y = 0$ to guarantee that they will hold in a family of hypersurfaces $y = const$.

Now, by simple algebraic manipulation equation (2) can be put in the canonical form

$$\frac{\partial^2 \bar{g}_{ij}}{\partial y^2} = F_{ij}\left(\bar{g}_{lm}, \frac{\partial \bar{g}_{lm}}{\partial y}\right), \tag{6}$$

where F_{ij} are analytical functions of their arguments. Therefore, according to the Cauchy-Kowalewski theorem [30], there exists a unique analytical solution $\bar{g}_{ik}(x, y)$ satisfying the analytical initial conditions :

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[1]In this letter we are adopting the following conventions for the Riemann and Ricci tensors, respectively: $R^\mu{}_{\nu\alpha\beta} = \Gamma^\mu{}_{\nu\alpha,\beta} - \Gamma^\mu{}_{\nu\beta,\alpha} + \Gamma^\tau{}_{\nu\alpha}\Gamma^\mu{}_{\tau\beta} - \Gamma^\tau{}_{\nu\beta}\Gamma^\mu{}_{\tau\alpha}$, $R_{\nu\beta} = R^\alpha{}_{\nu\alpha\beta}$.

$$\bar{g}_{ij}(x,0) = g_{ij}(x) \tag{7}$$

$$\left.\frac{\partial \bar{g}_{ij}}{\partial y}\right|_{y=0} = -2K_{ij}(x). \tag{8}$$

From the perspective of the embedding problem these initial conditions represent the metric and the extrinsic curvature of the hypersurface $y = 0$, whereas the solution of equation (2) gives the metric of the $(n+1)-$dimensional space. Thus, if we can guarantee that the constraint equations always have a solution, whatever the given metric g_{ik} is, then the theorem is proved, since the solution found $\bar{g}_{ij}(x,y)$ will satisfy $R_{\mu\nu} = 0$. Clearly, the embedding map is given by equation the $y = 0$.

It turns out, as Magaard has demonstrated [25], that the constraint equations always have a solution. Indeed, by simple counting operation we can see that there are $n(n+1)/2$ unknown functions (the algebraic independent elements of extrinsic curvature) and $n + 1$ constraint equations, since the metric $g_{ij}(x)$ must be considered as a given datum. For $n > 2$, there are more variables than equations. Thus using equation (4) to express one element of K_{ij} in terms of the others, Magaard has shown that equation (3) can be put in a canonical form with respect to n components of K_{ij} conveniently chosen. And then, once more, the Cauchy-Kowalewski theorem ensures the existence of the solution. It is important to note that, as we have said before, the number of variables is greater than the number of equations; in this sense, we can say that there are $(n+1)(n/2-1)$ degrees of freedom left over.

Now it happens that when the $(n+1)-$dimensional embedding space is sourced by a distribution of matter concentrated in a $n-$dimensional hypersurface (thin shell), then the extrinsic curvature of the hypersurface is not continuous. Following two different (although equivalent [31]) approaches, namely, the Israel-Darmois formalism [33,32] and the distributional method [34–36], it can be shown that the discontinuity of the extrinsic curvature is proportional to the energy-momentum tensor of the hypersurface. In Gaussian coordinates, we have

$$[K_{ij}] = \kappa\left(S_{ij} + \frac{g_{ij}}{1-n}S\right), \tag{9}$$

5

where κ is the gravitational constant in $(n+1)$-dimensions, $[K_{ij}] = \lim\limits_{y \to 0^+} K_{ij} - \lim\limits_{y \to 0^-} K_{ij}$, S_{ij} is energy-momentum tensor of the hypersurface and S is its trace. (The above equation is known as Lanczos equation [31].)

It is clear that when there is a discontinuity of the extrinsic curvature two different embeddings, one for each side of the hypersurface, are required. As we have mentioned earlier, the extrinsic curvature considered as part of the initial conditions in the scheme outlined above has some degrees of freedom left over. If $n = 4$, there are five completely independent components of the extrinsic curvature tensor. Each choice of the these components possibly determines different embedding spaces. The question now arises whether we can select two sets of these independent components in such a way that Lanczos equation may be satisfied for any S_{ij} .

Suppose we want to embed a n-dimensional spacetime with a given metric g_{ij} and a specific energy-momentum tensor S_{ij} in a $(n+1)$-dimensional vacuum space. Let $g_{ij}^-(x,y)$ be the metric of the embedding space determined by solving the dynamical equation (2) and imposing the initial conditions data: the metric of spacetime g_{ij} and the extrinsic curvature K_{ij}^-. Of course g_{ij} and K_{ij}^- must satisfy the constraint equations (3) and (4).

A new metric $g_{ij}^+(x,y)$ could be obtained by taking another extrinsic curvature K_{ij}^+ as initial data. Based on the Lanczos equations, let us define K_{ij}^+ by the expression

$$K_{ij}^+ = K_{ij}^- + \kappa \left(S_{ij} + \frac{g_{ij}}{1-n} S \right) \tag{10}$$

In order to be eligible for initial condition, K_{ij}^+ must satisfy the constraint equations. As a consequence, the following conditions must be imposed on S_{ik} and K_{ij}^- :

$$\nabla_i S_j^i = 0 \tag{11}$$

$$K_{ij}^- S^{ij} = -\frac{\kappa}{2} \left(S_{ij} S^{ij} + \frac{1}{1-n} S^2 \right). \tag{12}$$

These equations imply that S_{ij} and g_{ij} are not completely independent. The first equation, which does not involve K_{ij}^-, requires that S_{ij} be conserved with respect to the spacetime metric g_{ij}, which seems to be a reasonable condition. The second equation establishes an

additional constraint equation for the initial condition K_{ij}^-. As the extrinsic curvature has extra degrees of freedom (five, in the case of $n = 4$), this new constraint can, in principle, be solved for any g_{ij} and S_{ij} chosen. We should emphasize here that g_{ij}^- and g_{ij}^+ are analytical vacuum metrics defined in a five-dimensional open set surrounding the spacetime. The metric $\overline{g}_{ij}$ of the embedding space will be given by a combination of the two metrics g_{ij}^- and g_{ij}^+ :

$$\overline{g}_{ij}(x,y) = \begin{cases} g_{ij}^+(x,y) & y \geq 0 \\ g_{ij}^-(x,y) & y \leq 0. \end{cases} \tag{13}$$

The metric is continuous but its first derivative with respect to y has a discontinuity at $y = 0$. As the Cauchy-Kowalewski theorem ensures the analyticity of the solutions g_{ij}^- and g_{ij}^+, it follows that $\overline{g}_{ij}$ is piecewise analytical. Although g_{ij}^- and g_{ij}^+ are vacuum metrics, $\overline{g}_{ij}$ is generated by a Dirac delta source, i.e.

$$G_{\mu\nu} = -\kappa \delta(y) S_{\mu\nu}$$

where the components of $S_{\mu\nu}$ involving the extra coordinate are null.

It is remarkable that in this scheme the content of energy of the spacetime represented by the energy-momentum tensor S_{ij} does not necessarily determine the curvature of spacetime. Indeed, as we have seen, the only required relation between S_{ij} and g_{ij} is the conservation equation. This fact allows for very unconventional situations to take place. As we will see next, it is possible to have Minkowski spacetime filled with dust.

Consider the five-dimensional space metric given by

$$^{(5)}ds^2 = \begin{cases} -(a^+y+1)^2 dt^2 + dl^2 + dy^2 & y \geq 0 \\ -(2a^-y+1)^{-1} dt^2 + (2a^-y+1) dl^2 + dy^2 & y \leq 0, \end{cases}$$

where dl^2 denotes the Euclidean spatial interval and a^+, a^- are constants. (Note that for $y \neq 0$, the above metric corresponds to that of a vacuum space). Of course, the embedding map $y = 0$ gives Minkowski spacetime. But if we calculate the extrinsic curvature, it is easy to see that $K_{ij}^- = -diag(a^-, a^-, a^-, a^-)$ and that $K_{tt}^+ = a^+$ is the only non-null term of K_{ij}^+. Now, if we take $a^+ = a^- = \frac{\kappa}{3}\rho$, it follows, from Lanczos equation, that $S_{tt} = \rho$ and the other

components are null. This means that the embedded Minkowski spacetime is full of dust with uniform density equal to ρ.

It is well known that embeddings can be used as a mechanism of matter generation. This constitutes perhaps a key point of the so-called space-time-matter theory (STM), also referred to as induced-matter-theory [11,37–40]. Here, in contrast, we are in the presence of an utterly new and perhaps unexpected situation, where the matter confined to the brane does not necessarily curve the spacetime if the latter is conceived as a four-dimensional hypersurface embedded in a five-dimensional space endowed with a metrical structure which is not of differential class C^1.

Let us now employ the scheme described above to discuss the embedding of the spacetime in a Randall-Sundrum type model. In this case the $(n+1)$-dimensional space is characterized by two essential properties: it possesses Z_2 symmetry and is sourced by a cosmological constant Λ in the bulk. Thus, the Einstein tensor of the higher-dimensional space should satisfy the equation

$$G_{\mu\nu} = \Lambda g_{\mu\nu} - \kappa S_{\mu\nu} \delta\left(y\right). \tag{14}$$

The Z_2 symmetry implies that the metric must obey the condition

$$g_{\mu\nu}\left(x,y\right) = g_{\mu\nu}\left(x,-y\right), \tag{15}$$

which imposes the following relation between the "up" and "down" extrinsic curvatures of the brane

$$K_{ij}^+ = -K_{ij}^-. \tag{16}$$

As a consequence, the extrinsic curvature is now completely determined by the Lanczos equations

$$K_{ij}^+ = -K_{ij}^- = \frac{\kappa}{2}\left(S_{ij} + \frac{g_{ij}}{1-n}S\right). \tag{17}$$

The metrics $g_{\mu\nu}^+$ and $g_{\mu\nu}^-$ (respectively associated with K_{ij}^+ and K_{ij}^- as far as the initial conditions are concerned) should now satisfy the equation

$$G_{\mu\nu}^{\pm} = \Lambda g_{\mu\nu}^{\pm}. \tag{18}$$

Using the same procedure outlined above employed to analyze the vacuum equations, we can show that the existence of $g_{\mu\nu}^{+}$ and $g_{\mu\nu}^{-}$ is guaranteed by the Cauchy-Kowalewski theorem provided that the extrinsic curvatures K_{ij}^{+} and K_{ij}^{-} and the spacetime metric g_{ij} satisfy the new constraint equations [27]:

$$\nabla_j \left(K^{ij} - g^{ij} K \right) = 0 \tag{19}$$

$$R + K^2 - K_{ij} K^{ij} = -2\Lambda. \tag{20}$$

Since the extrinsic curvature K^{ij} is given by (17), these constraint equations imply the following equations between S_{ij} and g_{ij} :

$$\nabla_i S_j^i = 0 \tag{21}$$

$$\frac{\kappa^2}{4} \left(S_{ij} S^{ij} + \frac{S^2}{1-n} \right) = 2\Lambda + R. \tag{22}$$

It is not difficult to realize that $g_{\mu\nu}^{+}$ and $g_{\mu\nu}^{-}$ satisfy the condition $g_{\mu\nu}^{+}(x,y) = g_{\mu\nu}^{-}(x,-y)$. Indeed, assuming that $g_{\mu\nu}^{-}(x,y)$ is a solution of equation (18) associated with the initial conditions (7),(8) for g_{ij} and K_{ij}^{-}, we can verify that $g_{\mu\nu}^{-}(x,-y)$ also satisfies (18). Of course, the initial conditions of the latter now refer to g_{ij} and $-\left(K_{ij}^{-} \right)$. Since the Cauchy-Kowalewski theorem states that the analytical solution satisfying the initial conditions is unique, then we can conclude that K_{ij}^{+}, satisfying (16), generates $g_{\mu\nu}^{+}(x,y)$ with the mentioned property, i.e. $g_{\mu\nu}^{+}(x,y) = g_{\mu\nu}^{-}(x,-y)$. Therefore, we conclude that the metric of the $(n+1)$-dimensional space defined by (13) automatically obeys the Z_2 symmetry and that the existence of the embedding depends on the constraint equation between S_{ij} and g_{ij}. Each pair (g_{ij}, S_{ij}) that satisfies the constraint equations (21), (22) can be embedded in a five-dimensional space with the Z_2 symmetry generated by (14). In this scenario, unless additional restrictions are imposed on the bulk, the constraint equations (21) and (22) seems to represent the only connection between the geometry and the matter configuration of the brane.

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